Linear programming

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Definition 1. A problem of *optimisation* is one in which one tries to maximise or minimise a certain quantity called the *objective*, which depends on a finite number of variables. These may be either independent or related to one another through some *constraints*.

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Definition 2. A *mathematical programme* is an optimisation problem in which the objective and the constraints are given as functions or mathematical relationship. In other words,

$$\begin{aligned} \text{optimise:} \quad & z = f(x_1, \ldots, x_n) \\ \text{subject to:} \quad & g_i(x_1, \ldots, x_n) \left\{ \begin{array}{c} \leq \\ = \\ \geq \end{array} \right\} b_i, \quad i = 1, \ldots, m \end{aligned}$$

Some constraints are explicitly stated as requirements, others are hidden conditions. These latter need to be pin-pointed through the study and understanding of the model and its inputs.

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Definition 3. A *linear programme* is a mathematical programme all the functions involved of which are linear. This means that,

$$\begin{aligned} f(x_1, \ldots, x_n) &= c_1 x_1 + \cdots + c_n x_n \\ g_i(x_1, \ldots, x_n) &= a_{i1} x_1 + \cdots + a_{in} x_n \end{aligned}$$

where $i = 1, \ldots, m$ and c_j and a_{ij}, $j = 1, \ldots, n$, are constants. If there is an additional restriction on the input variables that they be all integers, then the optimisation problem is called an *integer programme*. A mathematical programme which is not a linear programme is said to be *nonlinear*.

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Definition 4. A *quadratic programme* is a mathematical programme in which all the constraints are linear and the objective function is in quadratic form, which is in general,

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_i x_j + \sum_{i=1}^{n} d_i x_i$$

where c_{ij} and d_i are constants.

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Definition 5. A linear programme is said to be in *standard form* if all the constraints are equalities and if one feasible solution is known. In other words, our problem is now

$$\begin{aligned} \text{optimise:} \quad & z = \mathbf{c}^T \mathbf{x} \\ \text{subject to:} \quad & A\mathbf{x} = \mathbf{b} \\ \text{with:} \quad & \mathbf{x} \geq 0 \end{aligned}$$

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Definition 6. One may change any linear programme into the standard form by adding a *slack variable* to the left-hand side of a constraint of the form $\sum a_{ij} x_j \leq b_i$ to obtain

$$\sum_{j=1}^{n} a_{ij} x_j + x_{p_k} = b_i$$

where $p_k > n$ and $k = 1, 2, \ldots$. Similarly one may add a *surplus variable* to the right-hand side of a constraint of the form $\sum a_{ij} x_j \geq b_i$ to obtain $\sum a_{ij} x_j = b_i + x_{q_l}$

$$\sum_{j=1}^{n} a_{ij} x_j - x_{q_l} = b_i$$

where $q_l > n$ and $l = 1, 2, \ldots$. Next, all the slack and surplus variables are added to the objective function with zero coefficients. Then if we add an *artificial variable* to the left-hand side of each constraint where there is no slack variable, then the *initial feasible solution* is $\mathbf{x}_0 = \mathbf{b}$, where $\mathbf{x}$ is the vector of slack and artificial variables. The artificial variables are added to the objective function with a large negative coefficient $-M$.

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Definition 7. A set of n vectors of m dimensions $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is said to be *linearly dependent* if there exist some constants $\alpha_1, \ldots, \alpha_n$ not all of which are zero, such that

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n = \mathbf{0} \tag{1}$$

It is said to be *linearly independent* if the condition in Equation 1 implies $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

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Theorem 1. Consider a set of n vectors of m dimensions. If $n > m$, then the set is linearly dependent.

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Problem 1. Prove Theorem 1.

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Definition 8. A vector $\mathbf{v}$ is called a *convex combination* of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ if there exist some nonnegative constants $\beta_1, \ldots, \beta_n$, where

$$\beta_1 + \cdots + \beta_n = 1$$

such that

$$\mathbf{v} = \beta_1 \mathbf{v}_1 + \cdots + \beta_n \mathbf{v}_n$$

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Definition 9. A set of m-dimensional vectors is said to be *convex* if for any two vectors belonging to the set the line segment between them also belongs to the set.

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Theorem 2. All points on the line segment joining any two vectors may be expressed as a convex combination of the two vectors.

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Problem 2. Prove Theorem 2.

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Definition 10. A vector $\mathbf{v}$ is called an *extreme point* of a convex set if it can not be expressed as a convex combination of two other vectorsin the set.

In other words, an extreme point of a convex set K is a point x in K that cannot be written as $x = \theta y + (1 - \theta)z$ with $0 < \theta < 1$, y and z in K, and $y \neq z$, that is to say, an extreme point is a point which is not an *interior point* of any line segment belonging to K.
An equivalent definition of an extreme point is that x is an extreme point of a convex set K if $K \setminus \{x\}$ is convex.

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Definition 11. A *metric space* is a non-empty set X for which is defined a concept of distance. The *distance* d is called a *metric* on X, having such properties that, for any points x and y in X, we have $d(x, y) \geq 0$, and $d(x, y) = 0$ implies $x = y$; $d(x, y) = d(y, x)$; and $d(x, y) \leq d(x, z) + d(z, y)$.

Let X be a metric space with metric d, let A be a subset of X and let x be any point of X. Then the *distance from x to A* is defined as

$$d(x, A) = \inf \{d(x, a) : a \in A\}$$

whereas the *diameter of A* is defined as

$$d(A) = \sup \{d(a_1, a_2) : a_1 \text{ and } a_2 \in A\}$$

Then a set is said to be *bounded* if its diameter is finite.
Further, let x_0 be a point in X and r a positive real number. Then the *open sphere* $S_r(x_0)$ with *centre* x_0 and *radius* r is the subset of X defined by

$$S_r(x_0) = \{x : d(x, x_0) < r\}$$

A point x in X is called a *limit point* of A if each open sphere centred on x contains at least one point of A different from x. A subset F of X is said to be *closed* if it contains all its limit points.

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Definition 12. A *linear space*, aka a *vector space*, is a non-empty set L on which is defined two binary processes, say *addition* and *scalar multiplication*. Addition is defined such that for any x, y and z in L, then $x + y$ is again in L; $x + y = y + x$; $x + (y + z) = (x + y) + z$; there exists a unique *identity* element 0, aka *zero element* or the *origin*, such that $x + 0 = x$ for every x; and there exists a unique *inverse* element $-x$ for every x, such that $x + (-x) = 0$. Scalar multiplication is defined with regard to *scalars*, some instances of which are real and complex numbers, such that for any scalar α and any x and y in L, αx is again in L; $\alpha(x + y) = \alpha x + \alpha y$; $(\alpha + \beta)x = \alpha x + \beta x$; $(\alpha\beta)x = \alpha(\beta x)$; and $1x = x$, where 1 is the identity for scalar multiplication. A *normed linear space* is a linear space on which is defined a *norm*, that is a function which maps each element x in the space to a real number $\|x\|$ in such a manner that $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$; $\|x + y\| \leq \|x\| + \|y\|$; and $\|\alpha x\| = |\alpha| \, \|x\|$.

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Theorem 3. A normed linear space is a metric space.

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Problem 3. Prove Theorem 3.

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Theorem 4. Any vector in a closed and bounded convex set with a finite number of extreme points can be expressed as a convex combinations of the extreme points.

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Problem 4. Prove Theorem 4.

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Definition 13. For two vectors, that is points, x and y in $\mathbf{R}^n$, we write $\mathbf{x} \geq \mathbf{y}$ if and only if $x_i \geq y_i$ for all $1 \leq i \leq n$. A system of m weak linear inequalilties in n variables can be written as $A\mathbf{x} \geq \mathbf{b}$, where A is an $m \times n$ matrix. A fundamental question concerning such system is whether it is *consistent*, that is to say, whether there exists some $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$. A system may be *inconsistent*, or it may have a set of solutions which is *unbounded*. If we sketch our problem on a graph, we may see that it's solution set is *convex*.

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Theorem 5. The solution space of a set of simultaneous linear equations is a convex set the number of extreme points of which is finite.

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Problem 5. Prove Theorem 5.

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Theorem 6. Let S be the set of all feasible solutions to the linear programme in standard form in Definition 5, in other words, S is the set of all vectors $\mathbf{x}$ that satisfy $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$, where A is an $m \times n$ matrix. Then S is a convex set, and the number of its extreme points is finite. The objective function attains its optimum, provided that one exists, at an extreme point of S. If $m \leq n$, then the extreme points of S have at least $n - m$ zero components.

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Problem 6. Prove Theorem 6.

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Algorithm 1 *Procedure for finding basic feasible solutions.*

Input: $A\mathbf{x} = \mathbf{b}$, A is an $m \times n$ matrix, $m \leq n$, $\operatorname{rank} A = m$

$[\,\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n\,] \leftarrow A$
$(x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b}) \leftarrow (A\mathbf{x} = \mathbf{b})$
for $i = m+1$ to n **do**
  $x_i \leftarrow 0$
endfor
$(x_1, \ldots, x_n) \leftarrow$ **solve** $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$

Definition 14. The *simplex method* is a matrix procedure which solves linear programmes of the standard form as described in Definition 5 where $\mathbf{b} > \mathbf{0}$. Starting from a basic feasible solution $\mathbf{x}_0$ we locate successively other basic feasible solutions giving better values for our objective. For minimisation programmes the method uses Table 1, for maximisation programmes the same table is also used but with the sign of entries in the bottom row reversed.

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Table 1 *Table used for minimisation programming in simplex method.*

		$\mathbf{x}^T$ $\mathbf{c}^T$	
$\mathbf{x}_0$	$\mathbf{c}_0$	A	$\mathbf{b}$
		$\mathbf{c}^T - \mathbf{c}_0^T A$	$-\mathbf{c}_0^T\mathbf{b}$

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Table 2 *Description of the simplex method.*

while negative number exists in **d do**

Locate the most negative number in the bottom row of the simplex table, excluding the last column. The column in which we find this number is called the *work column*. If more than one such column exist, choose one of them.

Find the smallest of the ratios between the elements in the last column and the elements in the work column of the same row, if these latter are positive. The element in the work column that yields this smallest ratio is called the *pivot element*. If there are more than one of these, choose one. If none of the elements in the work column is positive, the programme has no solution.

Using elementary row operations, convert the pivot element to 1 and reduce all other elements in the work column to 0.

Replace the x-variable in the pivot row and first column by the x-variable in the first row and pivot column. This new first column then becomes the current set of basic variables.

endwhile The optimal solution is one in which all the basic variables assume the corresponding values in the last column, while the remaining variables are zero. The optimal value of the objective function is then the value of the last row and last column for a maximisation programme, and the negative of this value if the programme is one of minimisation.

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Algorithm 2 gives a procedure for the simplex method. Here M represents a large positive integer, ρ a ratio, c a column, r a row, and $\mathbf{x}_0$ contains all the basic variables. Also the last row in Table 1 is represented here by $\mathbf{d} = \mathbf{c}^T - \mathbf{c}_0^T A$ and $e = -\mathbf{c}_0^T\mathbf{b}$.

Algorithm 2 *Algorithm for the simplex procedure.*

$j \leftarrow 0$
while there exists a negative number in **d do**
  $j \leftarrow j + 1$
  for $i = 1$ *to* n **do**
    $\{c\text{'s}\} \leftarrow$ (column number of the most negative number in the bottom row)
    (work column) $\leftarrow$ **choose** one of the $\{c\}$'s
    $k \leftarrow$ (work column)
  endfor
  $\rho_{pivot} \leftarrow M$
  $c \leftarrow 0$
  $soln \leftarrow 0$
  for $i = 1$ to m **do**

if $(\mathbf{a}_j)_{ik} > 0$ **then**
$soln \leftarrow 1$
$\rho \leftarrow \frac{(\mathbf{b}_j)_i}{(\mathbf{a}_j)_{ik}}$
if $\rho < \rho_{pivot}$ **then**
$r \leftarrow i$
endif
endif
endfor
if $soln = 0$ **then**
no solutions exist
endif
convert† A, such that $(\mathbf{a}_j)_{rk} = 1$ and $(\mathbf{a}_j)_{ik} = 0$, $1 \le i \le m$, $i \ne r$
$(\mathbf{x}_0)_r \leftarrow x_k$
endwhile
for $i = 1$ to m **do**
$(\mathbf{x}_0)_i^* \leftarrow (\mathbf{b}_j)_i$
endfor
for $i = m + 1$ to n **do**
$x_i^* \leftarrow 0$
endfor
$z^* \leftarrow e_j$
if the programme is one of minimisation **then**
$z^* \leftarrow -z^*$
endif

Definition 15. The *two-phase method* is a procedure modified from the simplex method to cope with cases when artificial variables exist in the initial solution $\mathbf{x}_0$, in order to minimise the round-off errors that occur in the calculation. The last row in Table 1 in this case is $\mathbf{d} = \mathbf{c}^T - \mathbf{c}_0^T A = \mathbf{d}_1 + M\,\mathbf{d}_2$, and consequently we have Table 3 which is used here. Algorithm 2 is then firstly applied to the last row, and then again to those elements directly above the zeros in that row. When an artificial variable is removed from the first column of the table, it ceases to be basic and may be removed from the top row of the table together with the entire column under it. When the last row contains only zeros, it may be deleted from the table. The programme has no solution if non-zero artificial variables are present in the final basic set.

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Table 1 *Table used for minimisation programming using the two-phase method.*

		$\mathbf{x}^T$ $\mathbf{c}^T$	
$\mathbf{x}_0$	$\mathbf{c}_0$	A	$\mathbf{b}$
		$\mathbf{d}_1$ $\mathbf{d}_2$	$-\mathbf{c}_0^T\mathbf{b}$

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Definition 16. Given a linear programme in the variables $x_1, \ldots, x_n$, there exists another linear programme associated with it, called its *dual*, which is in the variables $w_1, \ldots, w_m$. The original programme is called the *primal*. The primal completely determines the form of its dual. The *symmetric dual* of a primal linear programme in the matrix form

$$\begin{aligned} \text{minimise:} \quad & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to:} \quad & A\mathbf{x} \ge \mathbf{b} \\ \text{with:} \quad & \mathbf{x} \ge \mathbf{0} \end{aligned}$$

is the linear programme

$$\begin{aligned} \text{maximise:} \quad & z = \mathbf{b}^T\mathbf{w} \\ \text{subject to:} \quad & A^T\mathbf{w} \le \mathbf{c} \\ \text{with:} \quad & \mathbf{w} \ge \mathbf{0} \end{aligned}$$

† With the use of elementary row operations.

The dual variables $w_1, \ldots, w_m$ are known as *shadow costs*. The *unsymmetric dual* of the primal

$$\begin{aligned} \text{minimise:} \quad & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to:} \quad & A\mathbf{x} = \mathbf{b} \\ \text{with:} \quad & \mathbf{x} \geq 0 \end{aligned}$$

is

$$\begin{aligned} \text{maximise:} \quad & z = \mathbf{b}^T\mathbf{w} \\ & A^T\mathbf{w} \leq \mathbf{c} \end{aligned}$$

The unsymmetric dual of the primal

$$\begin{aligned} \text{maximise:} \quad & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to:} \quad & A\mathbf{x} = \mathbf{b} \\ \text{with:} \quad & \mathbf{x} \geq 0 \end{aligned}$$

is

$$\begin{aligned} \text{minimise:} \quad & z = \mathbf{b}^T\mathbf{w} \\ \text{subject to:} \quad & A^T\mathbf{w} \geq \mathbf{c} \end{aligned}$$

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Note 1. We may see from Definition 16 that the dual of a programme in standard form is not itself in standard form. These duals are said to be *unsymmetric*.

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Theorem 7. If an optimal solution exists for either the primal or the dual programme, then the other programme also has an optimal solution. If the duality is symmetric, then the two functions have the same optimal value. If the duality if unsymmetric, then the optimal value of each function can be derived from that of the other.

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